Geodezijske linije u R_n

Ako imamo zadatu krivu u R_n koja je data j-nama:

$$x^i = x^i(t), \qquad t_1 \le t \le t_2$$

t – parametar koji uzima vrednosti između t_1 i t_2.

Kvadrat elementa luka je:

$$ds^2 = g_{ij}\, dx^i dx^j = g_{ij}\, \dot{x}^i \dot{x}^j dt^2 \qquad (i, j = 1, \dots, n)$$

Dužina krive je:

$$s = \int_{t_1}^{t_2} \underbrace{\sqrt{g_{ij}\, \dot{x}^i \dot{x}^j}}_{F}\, dt$$

Ekstremala funkcionele $F = \sqrt{g_{ij}\, \dot{x}^i \dot{x}^j}$ određuje od svih mogućih rastojanja između te dve tačke ono minimalno.

Tačke linije nazivaju se geodezijske linije.

F zadovoljava Euler-Lagrangeovu dif. j-nu:

$$\frac{d}{dt}\frac{\partial F}{\partial \dot{x}^k} - \frac{\partial F}{\partial x^k} = 0$$

zavisi od koord. i od $\dot{x}$ u opštem slučaju

$$\frac{\partial F}{\partial \dot{x}^k} = \frac{1}{2}(g_{ij}\dot{x}^i\dot{x}^j)^{-1/2}\frac{\partial}{\partial \dot{x}^k}(g_{ij}\dot{x}^i\dot{x}^j) = \frac{1}{2}(g_{ij}\dot{x}^i\dot{x}^j)^{-1/2}\left[g_{ij}\frac{\partial \dot{x}^i}{\partial \dot{x}^k}\dot{x}^j + g_{ij}\dot{x}^i\frac{\partial \dot{x}^j}{\partial \dot{x}^k}\right]$$

$$= \frac{1}{2}(g_{ij}\dot{x}^i\dot{x}^j)^{-1/2}\Big[\underbrace{g_{ij}\delta^i_k\dot{x}^j}_{\underbrace{g_{jk}\dot{x}^j}_{j\to i}} + \underbrace{g_{ij}\delta^j_k\dot{x}^i}_{g_{ik}\dot{x}^i}\Big] = \frac{1}{2}(g_{ij}\dot{x}^i\dot{x}^j)^{-1/2}\, 2 g_{ik}\dot{x}^i$$

Pošto je: $\dfrac{ds}{dt} = \sqrt{g_{ij}\dot{x}^i\dot{x}^j} = \dot{s} \Rightarrow$ $\boxed{\dfrac{\partial F}{\partial \dot{x}^k} = \dfrac{1}{\dot{s}} g_{ik}\dot{x}^i}$

$$\boxed{\frac{\partial F}{\partial x^k} = \frac{1}{2}(g_{ij}\dot{x}^i\dot{x}^j)^{-1/2}\frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j = \frac{1}{2\dot{s}}\frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j}$$

$\dfrac{d}{dt}\left\{g_{ik}[x^j(t)]\right\}$ ПАЖЊ

$$\frac{d}{dt}\left[\frac{1}{\dot{s}}g_{ik}\dot{x}^i\right] - \frac{1}{2\dot{s}}\frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j = 0$$

$$\frac{\frac{d}{dt}(g_{ik}\dot{x}^i)\dot{s} - g_{ik}\dot{x}^i\ddot{s}}{\dot{s}^2} = \frac{\frac{\partial g_{ik}}{\partial x^j}\dot{x}^j\dot{x}^i\dot{s} + g_{ik}\ddot{x}^i\dot{s} - g_{ik}\dot{x}^i\ddot{s}}{\dot{s}^2} = \frac{\partial g_{ik}}{\partial x^j}\dot{x}^j\dot{x}^i\frac{1}{\dot{s}} + \frac{1}{\dot{s}}g_{ik}\ddot{x}^i - \frac{g_{ik}\dot{x}^i\ddot{s}}{\dot{s}^2}$$

ovo je sve

$$\Rightarrow \frac{\partial g_{ik}}{\partial x^j}\dot{x}^i\dot{x}^j - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j + g_{ik}\ddot{x}^i - \frac{g_{ik}\dot{x}^i\ddot{s}}{\dot{s}} = 0$$

17. III 1998.

22. XI '00

$$\frac{\partial g_{ik}}{\partial x^j}\dot{x}^i\dot{x}^j = \frac{1}{2}\frac{\partial g_{ik}}{\partial x^j}\dot{x}^i\dot{x}^j + \frac{1}{2}\underbrace{\frac{\partial g_{ik}}{\partial x^j}\dot{x}^i\dot{x}^j}_{\substack{i\to j\\ j\to i}} = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial x^j}\dot{x}^i\dot{x}^j + \frac{\partial g_{jk}}{\partial x^i}\dot{x}^j\dot{x}^i\right]$$

$$\Rightarrow \frac{1}{2}\left(\qquad\qquad\qquad\right)\dot{x}^i\dot{x}^j + g_{ik}\ddot{x}^i = \frac{g_{ik}\dot{x}^i\ddot{s}}{\dot{s}}$$

$$\Rightarrow \quad g_{ik}\ddot{x}^i + \Gamma_{ij,k}\dot{x}^i\dot{x}^j = \frac{g_{ik}\dot{x}^i\ddot{s}}{\dot{s}}$$

Ako za parametar t uzmemo dužinu luka s onda $\frac{ds}{dt}=1$, $\ddot{s}=0$

$$g_{ik}\frac{d^2x^i}{ds^2} + \Gamma_{ij,k}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \quad /\cdot g^{lk}$$

$$g^{lk}g_{ik}\frac{d^2x^i}{ds^2} = \delta^l_i\frac{d^2x^i}{ds^2} = \frac{d^2x^l}{ds^2} \quad ; \quad g^{lk}\Gamma_{ij,k} = \Gamma^l_{ij}$$

$$\Rightarrow \boxed{\frac{d^2x^l}{ds^2} + \Gamma^l_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0} \quad (i,j,l = 1,2,\dots,n)$$

to su dif-jne geodezijske linije

U slučaju E prostora sa metrikom $ds^2 = (dx^1)^2 + (dx^2)^2 + \dots + (dx^n)^2$

svi Γ simboli su nule $\Rightarrow \frac{d^2x^l}{ds^2} = 0$ tj. $x^l = a^l s + b^l$

što je j-na prave linije

13. XI '07.

Zamislite da hoćete da putujete iz Njujorka u Madrid, između dva grada na istoj geografskoj širini. Ako bi zemlja bila ravna najkraći put bi bio pravo na istok. Ako bi ste to učinili stigli bi ste u Madrid posle 5930 km. Ali zbog zakrivljenosti Zemlje, postoji i linija koja izgleda zakrivljeno, što će reći duže, na ravnoj mapi, iako je u stvarnosti kraća. Ako sledite geodezijsku liniju stići ćete na odredište posle 5768km pređenih kilometara. Prvo ćete ići na severoistok i polako skrenuti ka istoku, a zatim ka jugoistoku. Razlika je zbog zakrivljenosti zemlje, što je znak neeuklidske geometrije. Vazduhoplovne kompanije to dobro znaju.

KOVARIJANTNI I KONTRAVARIJANTNI IZVODI

Znamo da skup parc. izvoda $\frac{\partial \phi}{\partial x^i}$ -je kovarijantan vektor

Ali skup $\frac{\partial A_i}{\partial x^j}$ u opštem slučaju nije tenzor.

$$L_{ij} = \frac{\partial A_i}{\partial x^j}$$

$$\bar{L}_{ij} = \frac{\partial \bar{A}_i}{\partial \bar{x}^j} = \frac{\partial}{\partial \bar{x}^j}\left[\frac{\partial x^p}{\partial \bar{x}^i} A_p\right] =$$

$$= \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^i} A_p + \frac{\partial x^p}{\partial \bar{x}^i} \underbrace{\frac{\partial A_p}{\partial x^s}}_{L_{ps}} \frac{\partial x^s}{\partial \bar{x}^j}$$

$$= \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^i} A_p + \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^s}{\partial \bar{x}^j} L_{ps}$$

Da bismo ipak formirali tenzor izmnožimo A_i i $\underbrace{\frac{dx^i}{ds}}_{\text{kontrav. vektor}}$, tj. $A_i \frac{dx^i}{ds} \longrightarrow$ radi se o unutr. proizvodu (to je neki skalar) $\underline{\underline{\text{INV}}}$

Diferenciranjem po luku geod. linije dobićemo:

$$\frac{d}{ds}\left(A_i \frac{dx^i}{ds}\right) =$$

ili $$\frac{\partial A_i}{\partial x^j} \frac{dx^j}{ds} \frac{dx^i}{ds} + A_i \frac{d^2 x^i}{ds^2} =$$

iz dif. j-ne geod. linija

$$\Rightarrow A_i \frac{d^2 x^i}{ds^2} = A_e \frac{d^2 x^e}{ds^2} = - A_e \Gamma^e_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds}$$

$$\Rightarrow \underbrace{\frac{dx^i}{ds} \frac{dx^j}{ds}}_{\text{kontravar. tenzor}} \underbrace{\left(\frac{\partial A_i}{\partial x^j} - \Gamma^e_{ij} A_e\right)}_{\text{mora da bude kovarijantni tenzor prema zakonu količnika}} =$$

Obeležimo: $$A_{i,j} = \frac{\partial A_i}{\partial x^j} - \Gamma^e_{ij} A_e$$ indeks j nastaje diferenciranjem pa se odvaja zarezom.

Taj izraz se naziva kovarijantni izvod kovarijantnog vektora A_i po koordinati x^j

Potražimo sada analogni izraz ukoliko imamo kontravarijantni vektor A^k.

Prvo množenjem spuštamo index.

$$A_i = g_{ik} A^k$$

Onda: $$A_{i,j} = \frac{\partial}{\partial x^j}(g_{ik} A^k) - \Gamma^e_{ij} g_{ek} A^k = \frac{\partial g_{ik}}{\partial x^j} A^k + g_{ik} \frac{\partial A^k}{\partial x^j} - \Gamma^e_{ij} g_{ek} A^k$$

ili: $$A_{i,j} = g_{ik} \frac{\partial A^k}{\partial x^j} + A^k \left(\frac{\partial g_{ik}}{\partial x^j} - \Gamma^e_{ij} g_{ek}\right)$$

Sredimo izraz u zagradi:

$$\frac{\partial g_{ik}}{\partial x^j} - \Gamma^l_{ij} g_{lk} = \frac{\partial g_{ik}}{\partial x^j} - \underbrace{g^{lm}\Gamma_{ij,m} g_{lk}} =$$

$$= \frac{\partial g_{ik}}{\partial x^j} - \delta^m_k \frac{1}{2}\left(\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{mi}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m}\right) =$$

$$= \frac{\partial g_{ik}}{\partial x^j} - \frac{1}{2}\frac{\partial g_{jk}}{\partial x^i} - \frac{1}{2}\frac{\partial g_{ki}}{\partial x^j} + \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k} = \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{ki}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i}\right) = \Gamma_{jk,i}$$

Dakle:
$$\underline{\frac{\partial g_{ik}}{\partial x^j} - \Gamma^l_{ij} g_{lk} = \Gamma_{jk,i}}$$

Onda imamo:

$$A_{i,j} = g_{ik}\frac{\partial A^k}{\partial x^j} + \Gamma_{jk,i} A^k \quad / g^{li}$$

$$g^{li} A_{i,j} = g^{li} g_{ik}\frac{\partial A^k}{\partial x^j} + g^{li}\Gamma_{jk,i} A^k = \delta^l_k \frac{\partial A^k}{\partial x^j} + \Gamma^l_{jk} A^k$$

tj.
$$\underbrace{\underbrace{g^{li}}_{\text{TENZOR}} \underbrace{A_{i,j}}_{\text{TENZOR}}}_{\Downarrow} = \frac{\partial A^l}{\partial x^j} + \Gamma^l_{jk} A^k$$

proizvod je mešoviti tenzor tipa T^l_j

Dakle: $A^l_{,j} = \frac{\partial A^l}{\partial x^j} + \Gamma^l_{jk} A^k$

Ako izvršimo smenu $l \to i$, $k \to l$

Dobijamo
$$\underline{A^i_{,j} = \frac{\partial A^i}{\partial x^j} + \Gamma^i_{jl} A^l}$$

Ovaj izraz se naziva kovarijantni izvod kontravarij. vectora A^i po koord. x^j.
U slučaju E_N-prostora ovi izvodi se svode na obične parc. izvode.
Pored kovarijantnih izvoda $A_{i,j}$ i $A^i_{,j}$ mogu da se koriste i sledeći oblici izvedeni iz ovih.

$$A_i^{,j} = g^{jk} A_{i,k} \;;\quad A^{i,j} = g^{jk} A^i_{,k} = g^{jk} g^{ih} A_{h,k}$$

Kovarijantnim diferenciranjem videli smo da od kovarijantnih i kontrav. vektora dobijamo tenzore drugog reda. Dakle povećao se red za jedan i to u oba slučaja broj kovarijantnih indeksa. Analogno može se izvršiti "proširenje" datog tenzora na novi tenzor čiji je red za jedan veći.

Neka je zadan tenzor u_{ij} za koji znamo zakon transformacije:

$$\bar u_{mn} = u_{ij}\frac{\partial x^i}{\partial \bar x^m}\frac{\partial x^j}{\partial \bar x^n}$$

Diferenciranjem ove j-ne po x^p dobićemo

$$\frac{\partial \bar u_{mn}}{\partial \bar x^p} = \frac{\partial u_{ij}}{\partial x^h}\frac{\partial x^h}{\partial \bar x^p}\frac{\partial x^i}{\partial \bar x^m}\frac{\partial x^j}{\partial \bar x^n} + u_{ij}\frac{\partial^2 x^i}{\partial \bar x^m \partial \bar x^p}\frac{\partial x^j}{\partial \bar x^n} + u_{ij}\frac{\partial x^i}{\partial \bar x^m}\frac{\partial^2 x^j}{\partial \bar x^n \partial \bar x^p}$$

Ako za druge izvode zamenimo: (VIDI R/31)

$$\frac{\partial^2 x^i}{\partial \bar x^m \partial \bar x^p} = \bar\Gamma^h_{mp}\frac{\partial x^i}{\partial \bar x^h} - \Gamma^i_{rs}\frac{\partial x^r}{\partial \bar x^m}\frac{\partial x^s}{\partial \bar x^p}$$

$$\frac{\partial^2 x^j}{\partial \bar x^n \partial \bar x^p} = \bar\Gamma^h_{np}\frac{\partial x^j}{\partial \bar x^h} - \Gamma^j_{rs}\frac{\partial x^r}{\partial \bar x^n}\frac{\partial x^s}{\partial \bar x^p}$$

$$\frac{\partial \bar u_{mn}}{\partial \bar x^p} = \overbrace{\frac{\partial u_{ij}}{\partial x^h}\frac{\partial x^h}{\partial \bar x^p}\frac{\partial x^i}{\partial \bar x^m}\frac{\partial x^j}{\partial \bar x^n}}^{\mathrm{I}} + u_{ij}\left(\bar\Gamma^h_{mp}\frac{\partial x^i}{\partial \bar x^h} - \Gamma^i_{rs}\frac{\partial x^r}{\partial \bar x^m}\frac{\partial x^s}{\partial \bar x^p}\right)\frac{\partial x^j}{\partial \bar x^n} +$$

$$+ u_{ij}\frac{\partial x^i}{\partial \bar x^m}\left(\bar\Gamma^h_{np}\frac{\partial x^j}{\partial \bar x^h} - \Gamma^j_{rs}\frac{\partial x^r}{\partial \bar x^n}\frac{\partial x^s}{\partial \bar x^p}\right)$$

$$\frac{\partial \bar u_{mn}}{\partial \bar x^p} = \mathrm{I} + \bar u_{hn}\bar\Gamma^h_{mp} + \bar u_{mh}\bar\Gamma^h_{np} - \underbrace{u_{ij}\Gamma^i_{rs}\frac{\partial x^r}{\partial \bar x^m}\frac{\partial x^s}{\partial \bar x^p}\frac{\partial x^j}{\partial \bar x^n}}_{r\to i,\; s\to h,\; i\to r}$$

$$- \underbrace{u_{ij}\Gamma^j_{rs}\frac{\partial x^j}{\partial \bar x^m}\frac{\partial x^r}{\partial \bar x^n}\frac{\partial x^s}{\partial \bar x^p}}_{j\to r,\; r\to j,\; s\to h}$$

$$\frac{\partial \bar u_{mn}}{\partial \bar x^p} - \bar u_{hn}\bar\Gamma^h_{mp} - \bar u_{mh}\bar\Gamma^h_{np} = \frac{\partial u_{ij}}{\partial x^h}\frac{\partial x^h}{\partial \bar x^p}\frac{\partial x^i}{\partial \bar x^m}\frac{\partial x^j}{\partial \bar x^n} - u_{rj}\Gamma^r_{ih}\frac{\partial x^i}{\partial \bar x^m}\frac{\partial x^h}{\partial \bar x^p}\frac{\partial x^j}{\partial \bar x^n}$$

$$- u_{ir}\Gamma^r_{jh}\frac{\partial x^j}{\partial \bar x^n}\frac{\partial x^r}{\partial \bar x^m}\frac{\partial x^h}{\partial \bar x^p}$$

$$\frac{\partial \bar u_{mn}}{\partial \bar x^p} - \bar u_{hn}\bar\Gamma^h_{mp} - \bar u_{mh}\bar\Gamma^h_{np} = \left(\frac{\partial u_{ij}}{\partial x^h} - u_{rj}\Gamma^r_{ih} - u_{ir}\Gamma^r_{jh}\right)\frac{\partial x^j}{\partial \bar x^n}\frac{\partial x^r}{\partial \bar x^m}\frac{\partial x^h}{\partial \bar x^p}$$

Veličina $\frac{\partial u_{ij}}{\partial x^h} - u_{rj}\Gamma^r_{ih} - u_{ir}\Gamma^r_{jh} = u_{ij,h}$ se transformiše kao kov. tenzor trećeg reda. Ovaj tenzor se naziva kovarijantni izvod tenzora

$$\bar{\Gamma}_{ij,k} = \Gamma_{rs,p}\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}\frac{\partial x^p}{\partial \bar{x}^k} + g_{rs}\frac{\partial^2 x^r}{\partial \bar{x}^i \partial \bar{x}^j}\frac{\partial x^s}{\partial \bar{x}^k} \Big/ g^{mh}\frac{\partial \bar{x}^k}{\partial x^h}$$

$$g^{mh}\frac{\partial \bar{x}^k}{\partial x^h}\bar{\Gamma}_{ij,k} = \Gamma_{rs,p}\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}\underbrace{\frac{\partial x^p}{\partial \bar{x}^k} g^{mh}\frac{\partial \bar{x}^k}{\partial x^h}}_{\delta^p_h} + g_{rs} g^{mh}\frac{\partial^2 x^r}{\partial \bar{x}^i \partial \bar{x}^j}\underbrace{\frac{\partial x^s}{\partial \bar{x}^k}\frac{\partial \bar{x}^k}{\partial x^h}}_{\delta^s_h}$$

$$\delta^p_h g^{mh} = g^{mp} \qquad\qquad g_{rh}$$

Znamo:

$$\bar{g}^{lk} = \frac{\partial \bar{x}^l}{\partial x^n}\frac{\partial \bar{x}^k}{\partial x^h} g^{nh} \Big/ \frac{\partial x^m}{\partial \bar{x}^l}$$

$$\frac{\partial x^m}{\partial \bar{x}^l}\bar{g}^{lk} = \delta^m_n \frac{\partial \bar{x}^k}{\partial x^h} g^{nh}$$

$$\frac{\partial x^m}{\partial \bar{x}^l}\bar{g}^{lk} = g^{mh}\frac{\partial \bar{x}^k}{\partial x^h}$$

$$g^{mp}\Gamma_{rs,p}\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j} \qquad \underbrace{g_{rh} g^{mh}}_{\delta^m_r}\frac{\partial^2 x^r}{\partial \bar{x}^i \partial \bar{x}^j}$$

$$\Gamma^m_{rs}\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j} \qquad \delta^m_r \frac{\partial^2 x^r}{\partial \bar{x}^i \partial \bar{x}^j}$$

$$\frac{\partial x^m}{\partial \bar{x}^l}\bar{\Gamma}^l_{ij} = \Gamma^m_{rs}\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j} + \frac{\partial^2 x^m}{\partial \bar{x}^i \partial \bar{x}^j}$$

Dakle:

$$\frac{\partial^2 x^m}{\partial \bar{x}^i \partial \bar{x}^j} = \bar{\Gamma}^l_{ij}\frac{\partial x^m}{\partial \bar{x}^l} - \Gamma^m_{rs}\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}$$

Istim postupkom pokazuje se da kov. izvod u^{ij} po x^h je:

$$u^{ij}_{,h} = \frac{\partial u^{ij}}{\partial x^h} + u^{rj}\Gamma^{i}_{rh} + u^{ir}\Gamma^{j}_{rh}$$

Slično za kovarijantni izvod mešovitog tenzora u^i_j po x^h

$$u^i_{j,h} = \frac{\partial u^i_j}{\partial x^h} + u^r_j\Gamma^{i}_{rh} - u^i_r\Gamma^{r}_{jh}$$

Ili uopšteno za proizvoljni tenzor $U^{i_1 i_2 \cdots i_m}_{j_1 j_2 \cdots j_n}$ po x^k

$$U^{i_1 i_2 \cdots i_m}_{j_1 j_2 \cdots j_n, k} = \frac{\partial}{\partial x^k} U^{i_1 i_2 \cdots i_m}_{j_1 j_2 \cdots j_n} + U^{r i_2 \cdots i_m}_{j_1 j_2 \cdots j_n}\Gamma^{i_1}_{rk} + U^{i_1 r \cdots i_m}_{j_1 j_2 \cdots j_n}\Gamma^{i_2}_{rk} + \dots + U^{i_1 i_2 \cdots r}_{j_1 j_2 \cdots j_n}\Gamma^{i_m}_{rk}$$

$$- U^{i_1 i_2 \cdots i_m}_{r j_2 \cdots j_n}\Gamma^{r}_{j_1 k} - U^{i_1 i_2 \cdots i_m}_{j_1 r \cdots j_n}\Gamma^{r}_{j_2 k} - \dots - U^{i_1 i_2 \cdots i_m}_{j_1 j_2 \cdots r}\Gamma^{r}_{j_n k}$$

[USA 51] p. 89.

Primetimo da kovarijantni izvod Kronecker-ove delte je nula, odnosno da se ponaša kao konstanta

$$\delta^i_{j,e} = \frac{\partial \delta^i_j}{\partial x^e} - \Gamma^{\alpha}_{je}\delta^i_\alpha + \Gamma^{i}_{\alpha e}\delta^\alpha_j = 0 - \Gamma^{i}_{je} + \Gamma^{i}_{je} = 0$$

Ricci-jeva teorema: Kovarijantni izvod fundamentalnog tenzor je l

$$g_{ij,e} = \frac{\partial g_{ij}}{\partial x^e} - g_{\alpha j}\Gamma^{\alpha}_{ie} - g_{i\alpha}\Gamma^{\alpha}_{je}$$

$$\frac{\partial g_{ij}}{\partial x^e} = \Gamma_{ie,j} + \Gamma_{je,i}$$ sa strane: R/29

$$g_{ij,e} = \Gamma_{ie,j} + \Gamma_{je,i} - \Gamma_{ie,j} - \Gamma_{je,i} = 0$$

Kovarijantna formulacija prostornih izvoda

Pod gradijentom podrazumeva se kovarijantno diferenciranje
Ako je Φ - skalar, gradijent skalara Φ je skup parcijalnih izvoda $\Phi_{,i}$

$$(\mathrm{grad}\,\phi)_i = \phi_{,i} = \frac{\partial\phi}{\partial x^i}$$

Dakle, gradijent skalara je kovarijantni vektor.

Divergencija kontravarijantnog vektora A^i definiše se relacijom

$$\mathrm{div}\,A^i = A^i_{,i}$$

ili

$$\mathrm{div}\,A^i = \frac{\partial A^i}{\partial x^i} + \Gamma^i_{ie}A^e = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}A^i)$$

Rotor kovarijantnog vektora A_i definiše se razlikom kovarijantnih izvoda

$$(\mathrm{rot}\,A_i)_j = A_{i;j} - A_{j;i}$$

ili

$$(\mathrm{rot}\,A_i)_j = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}$$

(iskorišćena je simetričnost Cristoffel-ovih simbola.

Laplasijan skalara definišemo na sledeći način

$$\Delta\Phi = \mathrm{div}(g^{ij}\Phi_{,j}) = \frac{\partial}{\partial x^i}\left(g^{ij}\frac{\partial\phi}{\partial x^j} + \Gamma^i_{ie}\,g^{ej}\frac{\partial\phi}{\partial x^j}\right)$$

zamijeniti

Apsolutni (Bianchi-ev) izvod

[Intrinsic derivative p. 132]

R/34

Neka je $x^i = x^i(t)$ - ~~neka~~ kriva u R_n

Vektor $\frac{\delta A^\alpha}{\delta t}$ definisan formulom

$$\frac{\delta A^\alpha}{\delta t} \equiv \frac{dA^\alpha}{dt} + \Gamma^\alpha_{ij} A^i \frac{dx^j}{dt}, \qquad (\alpha = 1, 2 \ldots)$$

naziva se apsolutni (Bianchi-ev) izvod kontravarijantnog vektora A^α po parametru t. Analogno može da se definiše aps. izvod kovar. vektr.

$$\frac{\delta A_\alpha}{\delta t} = \frac{dA_\alpha}{dt} - \Gamma^\varrho_{\alpha j} \frac{dx^j}{dt} A_\varrho$$